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ABSTRACT WEINGARTEN SURFACES

TILLA KLOTZ MILNOR

1. Introduction

Suppose a pair of real quadratic forms A and B is prescribed on an oriented surface S. If A is definite, one can imitate many classical procedures involving the fundamental forms I and II on a surface in 3-space. In particular, there are obvious analogs $H = H(A, B)$ and $K = K(A, B)$ of mean and extrinsic curvature, and one easily defines sequences of fundamental forms $X_n = X_n(A, B)$ and skew fundamental forms $X_n' = X_n'(A, B)$ (See §2.) If some nontrivial equation is satisfied on S connecting H, K, and (perhaps) the intrinsic curvatures of certain of the forms X_n or X_n', we call S an abstract Weingarten surface.

It is no surprise that various results from the theory of immersed surfaces can be recaptured in this setting. Indeed, a good deal of literature is based, to one extent or another, on this realization. References [7], [8], [25], [26], [27] and [28] provide just a few examples.

In this paper, we give abstract versions of some simple theorems from surface theory. In particular, we study the situation in which H and K satisfy a linear equation. We describe the exact connection between the Codazzi-Mainardi equations and the appearance in seemingly unrelated situations (see Examples 1 through 5 in §3) of certain holomorphic quadratic differentials. The Main Lemma is independent of the Codazzi-Mainardi equations, and gives information deduced from the one assumption that a particular quadratic differential associated with B is holomorphic on the conformal structure determined by A.

The usefulness of most results below depends upon the identification in natural geometric settings of pairs A, B which satisfy their hypothesis. Such applications are provided by Theorems 1 and 2, the Corollary to Theorem 3, and Example 3. In addition, the Main Lemma, Corollary 2 to Lemma 1, and Theorem 3 have already proved valuable in the study of harmonically immersed surfaces. (See [19], [20] and [22].)

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2. Preliminary notions

Throughout this paper, we assume C^∞ smoothness, and we use the symbols α, β, γ and c to denote constants. Let S be an oriented surface. (Otherwise work with its universal cover.) Real quadratic forms $A = Edx^2 + 2Fdxdy + Gdy^2$ and $B = Ldx^2 + 2Mdxdy + Ndy^2$ on S yield *a pair* A, B if A is nondegenerate. It is convenient to use the terminology applicable when $A = I$ and $B = II$ are the fundamental forms of classical surface theory. Thus we call a pair A, B *flat* if $B \equiv 0$ and *spherical* if $B \equiv cA \neq 0$. A point on S where $B \alpha A$ is called an *umbilic*. Define the *mean* and *extrinsic curvatures* H and K of a pair A, B by

$$H = H(A, B) = \frac{EN + GL - 2FM}{2(EG - F^2)}, \qquad K = K(A, B) = \frac{LN - M^2}{EG - F^2}.$$

If A is definite, call the pair A, B *fundamental*, define the *skew curvature* H' by

$$H' = H'(A, B) = (H^2 - K)^{\frac{1}{2}},$$

and denote by k_1 and k_2 the *principal curvatures* $H \pm H'$ taken in whichever order is convenient. Given a fundamental pair A, B

$$2H = k_1 + k_2, \quad 2H' = |k_2 - k_1|, \quad K = k_1 k_2,$$

so that umbilics are characterized by $H' = 0$. Moreover

$$H(B, A)K(A, B) = H(A, B), \quad K(A, B)K(B, A) = 1, \tag{1}$$

if B is nondegenerate. When B, A is also a fundamental pair,

$$\begin{aligned} H'(B, A)|K(A, B)| &= H'(A, B), \\ |H(A, B)|H'(B, A) &= |H(B, A)|H'(A, B). \end{aligned} \tag{2}$$

Wherever $H' \neq 0$ and throughout the interior of the umbilic set, there are local coordinates x, y orthogonal for both A and B, so that

$$A = Edx^2 + Gdy^2, \quad B = k_1 Edx^2 + k_2 Gdy^2.$$

Such *doubly orthogonal* coordinates are locally available for a fundamental pair on an open dense subset of S. It is therefore convenient to use them to check (by continuity) identities valid on all of S.

If a pair A, B is given, call coordinates x, y *asymptotic* if $L \equiv N \equiv 0$, and *Tchebychev* if $|E| = |G| = 1$. Let $K(A)$ denote the intrinsic curvature of A, so that, when $A = Edx^2 + Gdy^2$,

$$4(EG)^2 K(A) = G_x(EG)_x + E_y(EG)_y - 2EG(G_{xx} + Eyy). \tag{3}$$

If $E = G = \lambda$ above,

(4) $$-2\lambda K(A) = \Delta \log|\lambda| = (\log \lambda)_{xx} + (\log \lambda)_{yy}.$$

Here $\lambda \neq 0$ because A is nondegenerate.

To any fundamental pair A, B we associate doubly infinite sequences X_n and X_n' of *fundamental* and *skew fundamental forms*, as follows. (See [16] or [17].)

Let $X_1 = A, X_2 = B$ and set

(5) $$X_n = 2HX_{n-1} - KX_{n-2}$$

for $n \geqslant 3$. Wherever $K \neq 0$, use (5) to give X_n for $n \leqslant 0$ as well. Wherever $H' \neq 0$,

(6) $$H'X_n' = X_{n+1} - HX_n$$

defines X_n' for $n \geqslant 1$, and for $n \leqslant 0$ wherever $H'K \neq 0$. *In statements involving a form X_n or X_n' any condition necessary for its definition is implicitly assumed.* In terms of doubly orthogonal coordinates.

(7) $$X_n = k_1^{n-1}Edx^2 + k_2^{n-1}Gdy^2, \quad \pm X_n' = k_1^{n-1}Edx^2 - k_2^{n-1}Gdy^2,$$

where $\pm$ is the sign of $k_1 - k_2$. The superscript $^{(\prime)}$ wll be used when the prime in parenthesis can be consistently included or excluded. Where $X_j^{(\prime)}$ is definite,

(8) $$\begin{aligned} &H\left(X^{(\prime)}_j, X^{(\prime)}_{j+1}\right) = H, \quad K\left(X^{(\prime)}_j, X^{(\prime)}_{j+k}\right) = K, \\ &H\left(X^{(\prime)}_j, X^{(\prime)}_{j-1}\right) = H/K, \quad K\left(X^{(\prime)}_j, X^{(\prime)}_{j-1}\right) = 1/K. \end{aligned}$$

If we set $A' = X_1'(A, B)$ and $B' = X_2'(A, B)$, we get a fundamental pair A', B' with the same H and K as A, B. Thus

(5′) $$X_n' = 2HX_{n-1}' - KX_{n-2}'$$

and

$$X_n'(A, B) = X_n(A', B'), \quad X_n'(A', B') = X_n(A, B).$$

By (7), the $X_n^{(\prime)}$ are definite (or indefinite) for all odd n if $A^{(\prime)}$ is definite (or indefinite). If X_n is definite, then X_n' is indefinite, and vice versa. A shift from A, B to a fundamental pair X_j, X_{j+1} shifts each $X_n^{(\prime)}$ back $j - 1$ places. Thus

$$X^{(\prime)}_n(X_j, X_{j+1}) = X^{(\prime)}_{n+1-j}$$

and similarly

(9) $$X^{(\prime)}_n(X_j, X_{j-1}) = X^{(\prime)}_{j+1-n}.$$

In classical notation, $III = X_3(I, II)$. We set $C^{(\prime)} = X_3^{(\prime)}(A, B)$. By (9),

$$X^{(\prime)}_3(C, B) = A^{(\prime)},$$

so that, in the classical situation, $I = X_3(III, II)$.

The forms $X_n^{(\cdot)}$ by no means exhaust the supply of real quadratic forms on S which can be associated with a fundamental pair A, B. Consider, for example, the "*lines-of-curvature*" *form* $W = W(A, B)$ given by

$$(10)\qquad \begin{aligned}&(EG - F^2)^{\frac{1}{2}} W \\ &\qquad = (EM - FL)dx^2 + (EN - GL)dxdy + (FN - GM)dy^2.\end{aligned}$$

(See [11].) Integral curves for the equation $W = 0$ are level lines for doubly orthogonal coordinates. Umbilics are points where $W \equiv 0$ for all values of dx and dy. Since singularities in the net of curves satisfying $W = 0$ occur only at umbilics, one has the familiar fact that H' must vanish somewhere on any compact S of genus $g \neq 1$ for any fundamental pair A, B. Note that

$$(11)\qquad H(A, W(A, B)) = H(B, W(A, B)) = 0.$$

Other quadratic forms related to A, B are discussed in §3.

We call a pair A, B *Codazzi* and write Cod(A, B) in case

$$(12)\qquad \begin{aligned} L_y - M_x &= L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2, \\ M_y - N_x &= L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2, \end{aligned}$$

where the Christoffel symbols are computed for A. These equations take on the simple form

$$(13)\qquad L_y = E_y H, \quad N_x = G_x H,$$

or

$$(14)\qquad \mp (k_1)_y E = E_y H', \quad \pm (k_2)_x G = G_x H',$$

for doubly orthogonal coordinates x, y. Here $\pm$ is the sign of $k_1 - k_2$, and $\mp$ its negative. Basic facts about Codazzi pairs are proved in [21]. We cite just the following.

Fact 1. A pair A, B is flat or spherical if Cod(A, B) with $B \alpha A$.

Fact 2. Suppose $K(A, B) \neq 0$ for a fundamental pair A, B. Then

(i) Cod(X_j, X_k) if and only if Cod(X_j', X_k'),

(ii) Cod(X_j, X_k') if and only if Cod(X_j', X_k), and

(iii) Cod$(X_j^{(\cdot)}, X_{j+k}^{(\cdot)})$ if and only if Cod$(X_{j+2k}^{(\cdot)}, X_{j+k}^{(\cdot)})$.

Fact 3. Suppose A, B is a fundamental pair. If Cod(A, B) and Cod$(\lambda A, B)$, then λ is constant near any point where $H \neq 0$. If Cod(A, B) and $H \equiv 0$, then Cod$(\lambda A, B)$ holds for all λ which are locally constant on the umbilic set.

Fact 4. Suppose A, B is a fundamental pair. If Cod(A, B) and Cod$(A, \lambda B)$, then λ is constant near any point where $K \neq 0$. If Cod(A, B) and $K \equiv 0$, then near any point where $H' \neq 0$, there is a nonconstant function λ such that Cod$(A, \lambda B)$.

Fact 5. If $\mathrm{Cod}(A, B)$ and $\mathrm{Cod}(B, A)$ for a fundamental pair A, B, then $K \equiv 0$ and $K(A) \equiv K(B) \equiv 0$ on the closure of the nonumbilic set.

Fact 6. If $\mathrm{Cod}(A, B)$ and $K(A, B) \neq 0$ for a fundamental pair A, B then $K(C) = K(A)/K(A, B)$. (See [16] or [28].)

Remark 1. Fact 6 shows that $\mathrm{Cod}(A, B)$ and $K(C) \equiv 1$ imply the theorem egregium equation $K(A) = K(A, B)$ whenever $K(A, B) \neq 0$ for a fundamental pair A, B. Similarly, by (8), if $K(I) \equiv 1$ on an S in E^3, S^3 or H^3, the theorem egregium equation $K(III) = K(III, II)$ holds for the Codazzi pair III, II wherever $K(I, II) \neq 0$. Thus there exists (locally) a not necessarily different immersion of S in E^3 achieving III as its first fundamental form and II as its second, wherever $K(I, II) \neq 0$.

Most results in [6] and [17] remain valid if the pair I, II is replaced by a fundamental Codazzi pair A, B. In particular, we have the following statement. (See [29].)

Fact 7. If $\alpha + \beta H + \gamma K \equiv 0$ for a fundamental Codazzi pair A, B, then $\alpha A + \beta B + \gamma C$ is flat wherever it is nondegenerate, that is, flat wherever $(\alpha + \beta k_1 + \gamma k_1^2)(\alpha + \beta k_2 + \gamma k_2^2) \neq 0$.

3. Fundamental pairs on a Riemann surface

By R we denote a Riemann surface (or conformal structure) on S. To work on R, use only those coordinates x, y on S for which $z = x + iy$ is a conformal parameter on R. If A is a definite real quadratic form on S, we write $R = R_A$ in case $A = \lambda dz d\bar{z}$ on R for some function λ. When $R = R_A$, A is called an R-conformal metric. Use of the symbol R_A automatically implies that A is definite.

Suppose now that $B = L dx^2 + 2M dx dy + N dy^2$ is any real quadratic form on S. Working on R, we define the forms Ω, Γ, Π and T by

$$
\begin{aligned}
4\omega = 4\Omega(B, R) &= (L - N - 2iM) dz^2, \\
2\Gamma(B, R) &= (L + N) dz d\bar{z}, \\
\Pi = \Pi(B, R) &= |LN - M^2|^{\frac{1}{2}} dz d\bar{z}, \\
T = T(B, R) &= N dx^2 - 2M dx dy - L dy^2.
\end{aligned} \tag{15}
$$

To the quadratic differential Ω, we associate the real quadratic forms $|\Omega|$, $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ on R. Note that

$$B = 2\,\mathrm{Re}\,\Omega + \Gamma = \Omega + \bar{\Omega} + \Gamma, \quad T = 2\Gamma - B. \tag{16}$$

Given a definite form B, $R = R_B$ if and only if $\Omega \equiv \bar{\Omega} \equiv 0$. Those familiar with Riemann surfaces witll recognize $2\bar{\Omega}/(\Gamma + \Pi)$ as the Beltrami differential on R associated with a positive definite B. (See [2].)

By the rules of tensor calculus, the forms Γ, Π, T, $|\Omega|$, Re Ω and Im Ω extend to real quadratic forms on S. In terms of *any* R-conformal metric A on S one has

$$(17)\quad \begin{array}{lll} \Gamma(B, R_A) = HA, & \Pi(B, R_A) = |K|^{\frac{1}{2}}A, & T(B, R_A) = 2HA - B \\ 2|\Omega(B, R_A)| = H'A, & 2\,\mathrm{Re}\,\Omega(B, R_A) = H'A', & 2\,\mathrm{Im}\,\Omega(B, R_A) = W, \end{array}$$

where H, K, H' and W are computed for the fundamental pair A, B. Using (15) and (16), it is easy to check that

$$(18)\quad \begin{array}{l} H(A, \mathrm{Re}\,\Omega) \equiv H(A, \mathrm{Im}\,\Omega) \equiv H'(A, \Pi) \equiv H'(A, \Gamma) \equiv H'(A, |\Omega|) \equiv 0, \\ H(A, T) = H, \quad K(A, \Pi) = |K|, \quad K(A, \mathrm{Re}\,\Omega) \leqslant 0, \quad K(A, \mathrm{Im}\,\Omega) \leqslant 0. \end{array}$$

If $\pm$ equals sign(det B) $\neq 0$, $X_3(B, \Pi) = \pm T$ and

$$(19)\quad \begin{array}{lll} H(B, \mathrm{Im}\,\Omega) = 0, & K(B, T) = 1, & K(\Pi, T) = K(B, \Pi) = \pm 1, \\ K(B, \mathrm{Im}\,\Omega) \leqslant 0, & H^2(B, T) \geqslant 1, & H^2(\Pi, T) \geqslant 1. \end{array}$$

If B is definite,

$$(20)\quad H(\Gamma, B) \equiv 1, \quad K(B, \Pi) \equiv 1, \quad 0 \leqslant K(\Gamma, B) \leqslant 1, \quad H^2(B, \Pi) \geqslant 1.$$

Lemma 1. *If A and B are positive definite and B is complete, then HA is complete.*

Proof. Work on R_A. By (15) and (17), $2\Gamma - B = N dx^2 - 2M dx dy + L dy^2$ is positive definite, since B is. Thus $B < 2\Gamma$, and since $\Gamma = HA$, the lemma follows.

Corollary 1 to Lemma 1. *Suppose A and B are positive definite with B complete. If H is bounded, A is complete. If H/K is bounded, C is complete.*

Proof. By (8), $H(C, B) = H/K$, so one need only apply Lemma 1 to the fundamental pairs A, B and C, B.

If A is positive definite, the energy function $e(f)$ of an immersion f: $(S, A) \to M^n$ of S in a Riemannian manifold M^n is given by $H = H(A, I)$, where I is the metric induced on S by f. (See [5].) Indeed, the immersion f: $(S, HA) \to M^n$ has energy $e(f) \equiv 1$, which yields the following.

Corollary 2 to Lemma 1. *For an immersion f: $(S, A) \to M^n$ with constant energy $e(f)$, A is complete if I is complete.*

Denote by $R_n^{(\prime)}$ the conformal structure determined on S by a definite form $X_n^{(\prime)}$. Lemma 3 in [15] provides considerable arithmetic information about the forms Ω and Γ associated with an $X_j^{(\prime)}$ on R_n or R_n'. In particular,

$$(21)\quad \Gamma(X_n', R_n) = \Gamma(X_n, R_n') = 0, \quad 2|\Omega(X_n', R_n)| = X_n', \; 2|\Omega(X_n, R_n')| = X_n.$$

Using (8), (15) and (17), one gets

$$(22)\quad \begin{array}{c} 2|\Omega(C, R_B)| = 2K|\Omega(A, R_B)| = H'B, \quad 2K|\Omega(B, R_C)| = H'C, \\ \Gamma(C, R_B) = K\Gamma(A, R_B) = HB, \quad K\Gamma(B, R_C) = HC, \end{array}$$

while by (5), (5′) and (6), one gets

$$(23)\quad \begin{gathered} \Omega(C, R_A) = 2H\Omega(B, R_A), \quad H'\Omega(A', R_A) = \Omega(B, R_A), \\ H'\Omega(A', R_B) = -H\Omega(A, R_B), \\ 2H\Omega(B, R_C) = K\Omega(A, R_C), \quad H'\Omega(B', R_C) = -H\Omega(B, R_C), \\ \Omega(C, R_B) = -K\Omega(A, R_B) = H'\Omega(B', R_B). \end{gathered}$$

A quadratic differential $\Omega = \phi dz^2$ on R is *holomorphic* if $\phi = \phi(z)$ is complex analytic for each conformal parameter z on R. Thus the zeros of a holomorphic Ω are isolated unless $\Omega \equiv 0$.

Formulas (22) and (23) remain valid if one simultaneously exchanges the symbols A and A', B and B', and C and C'. Many statements of the following sort can be deduced from (23). When $H \equiv c \neq 0$, $\Omega(B, R_A)$ is holomorphic if and only if $\Omega(C, R_A)$ is. When $K \equiv c \neq 0$, $\Omega(A, R_B)$ is holomorphic if and only if $\Omega(C, R_B)$ is.

We now describe some classical situations in which an Ω associated with $I^{(\prime)}$, $II^{(\prime)}$ or III is holomorphic. These examples motivated the study undertaken in this paper.

Example 1. (See [10].) $H \equiv c$ on a surface in E^3 if and only if $\Omega(II, R_1)$ is holomorphic.

Example 2. (See [12].) $K \equiv c > 0$ on a surface in E^3 if and only if $\Omega(I, R_2)$ is holomorphic.

Example 3. (See Lemma 9 in §5.) $K \equiv c < 0$ on a surface in E^3 if and only if $\Omega(I', R_2')$ is holomorphic.

Example 4. (See [4] and [18].) If f: $(S, A) \to M^n$ is a harmonic immersion of S in a Riemannian manifold M^n, then $\Omega = \Omega(I, R_A)$ is holomorphic. Indeed, when $n = 2$, f is harmonic if and only if Ω is holomorphic. Thus the search for Riemannian metrics A and B on S for which $\Omega(B, R_A)$ is holomorphic coincides with the search for A and B such that id: $(S, A) \to (S, B)$ is harmonic.

Example 5. (See [4] and [24].) $H \equiv c$ on a surface in E^3 if and only if $\Omega(III, R_1)$ is holomorphic (that is, if and only if the Gauss map is harmonic).

4. The main lemma and some applications

A great deal is implied by the simple assumption that $\Omega(B, R_A) \not\equiv 0$ is holomorphic, as the next lemma indicates. The result incorporates Lemmas 1 and 2 from [13] and Lemma 1 from [15], but much of it is new. See [1] for the definitions of subharmonic and superharmonic functions.

Main lemma. *Suppose $\Omega(B, R_A) \not\equiv 0$ is holomorphic, with A positive definite. Then except at isolated points where $H' = 0$, the following statements hold.*

(i) *$H'A$, $H'A'$ and W are flat,*

(ii) *There are coordinates x, y such that $H'A = dx^2 + dy^2$ and $H'B = (H + H')dx^2 + (H - H')dy^2$,*

(iii) *The function $\log H'$ is subharmonic on R_A if $K(A) \geqslant 0$ and superharmonic on R_A if $K(A) \leqslant 0$,*

(iv) *For positive definite B, $\cosh^{-1}(H/H') > 0$ is superharmonic on R_A if $K(B) \geqslant 0$, subharmonic on R_A if $K(B) \leqslant 0$, and constant only if $K(B) \equiv K(HA) \equiv 0$, while H/H' is itself subharmonic on R_A if $K(B) \leqslant 0$.*

(v) *For indefinite B, H/H' is subharmonic on R_A if $K(B) \geqslant 0$ and $H \leqslant 0$, superharmonic on R_A if $K(B) \leqslant 0$ and $H \geqslant 0$, and constant only if $K(B) \equiv K(HA) \equiv 0$.*

Moreover, everywhere on S these statements hold.

(vi) *For positive definite B, $K(B) \geqslant (H^2/K)K(HA)$, so that $K(B) \geqslant 0$ where $K(HA) \geqslant 0$, and $K(HA) \leqslant 0$ where $K(B) \leqslant 0$.*

(vii) *For indefinite B, $K(B) \leqslant 0$ where $K(HA) \geqslant 0$ and $H < 0$, while $K(B) \geqslant 0$ where $K(HA) \leqslant 0$ and $H > 0$.*

Proof. The isolated zeros of the holomorphic $\Omega = \Omega(B, R_A) \not\equiv 0$ occur where $H' = 0$. Wherever $H' \neq 0$, there are R_A conformal parameters $z = x + iy$ in terms of which $c\Omega = dz^2$ for any $c \neq 0$. (See [3].) Thus the metrics $|\Omega|$, $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ are flat wherever $H' \neq 0$, and part (i) follows by (17). If $c = 2$, one has $A = \lambda(dx^2 + dy^2)$ and $B = Ldx^2 + (L - 2)dy^2$. If A is positive definite, then $\lambda > 0$, $\lambda H = L - 1$, and $\lambda^2 K = L(L - 2)$. Thus $\lambda H' = 1$, $H'L = H + H'$, $H'(L - 2) = H - H'$, and part (iii) follows. Since $\lambda = 1/H'$ above, (4) yields

$$2K(A) = H'\Delta \log H',$$

which confirms part (iii). If $f = H/H'$, then (3) and (4) yield

$$2(f^2 - 1)K(B) + \Delta f = f\left(f_x^2 + f_y^2\right)/\left(f^2 - 1\right), \tag{24}$$

$$2f^2K(HA) + \Delta f = \left(f_x^2 + f_y^2\right)/f. \tag{25}$$

Subtraction of (25) from (24) gives

$$K(B) = H^2K(HA)/K + \left\{H'^5\left(f_x^2 + f_y^2\right)/2HK^2\right\}. \tag{26}$$

When A and B are positive definite, $f > 1$, so that part (iv) follows from (24). If A is positive definite and B indefinite, $f^2 < 1$ and $fH > 0$. Since $K = (H'k)^2(f^2 - 1)$, part (v) follows from (24). To check parts (vi) and (vii) where $H' \neq 0$, use (26). By continuity, the inequalities involved extend to all of S.

Corollary to the main lemma. *If* $\Omega(X_n, R_n')$ *or* $\Omega(X_n', R_n)$ *is holomorphic,* $X_n, X_n', W(X_n, X_n')$ *and* $W(X_n', X_n)$ *are flat while* $\mathrm{Cod}(X_n, X_n')$ *and* $\mathrm{Cod}(X_n', X_n)$.

Proof. By (7), $H(X_n, X_n') \equiv H(X_n', X_n) \equiv 0$ and $K(X_n, X_n') \equiv K(X_n', X_n) = -1$, so that $H'(X_n, X_n') \equiv H'(X_n', X_n) \equiv 1$. By part (i) of the lemma, $X_n, X_n', W(X_n, X_n')$ and $W(X_n', X_n)$ are flat. Indeed, using the coordinates provided by part (ii), X_n and X_n' have constant coefficients. Thus $\mathrm{Cod}(X_n, X_n')$ and $\mathrm{Cod}(X_n', X_n)$ hold.

In several proofs, we will use the fact (see [1]) that a subharmonic function bounded above (or a superharmonic function bounded below) must be constant on a parabolic Riemann surface. It follows that a subharmonic function bounded above (or a superharmonic function bounded below) by a harmonic function must itself be harmonic on a parabolic Riemann surface. The next result is closely related to material in [9], [13], [30] and [31].

Lemma 2. *Suppose that* A *is positive definite and complete, while* $\Omega = \Omega(B, R_A) \not\equiv 0$ *is holomorphic. Then* $K(A) \equiv 0$ *and* $H' \equiv c \neq 0$ *if either* $K(A) \geqslant 0$ *with* H' *bounded or* $K(A) \leqslant 0$ *with* $1/H'$ *bounded.*

Proof. Since $\Omega \not\equiv 0$ is holomorphic, R_A is not conformally the sphere. If A is a complete Riemannian metric with $K(A) \geqslant 0$, R_A is parabolic by Lemma 5 in [13]. If $K(A) \geqslant 0$ and $H' \leqslant c$, the Main Lemma (iii) states that $\log H' \leqslant \log c$ is subharmonic on R_A, except at isolated points where $H' \neq 0$. But isolated values of $-\infty$ for a subharmonic function are allowed. (See [1].) Thus $H' \neq 0$ is constant, and $K(A) \equiv 0$.

Remark 2. In [15], we replace the assumption that $\Omega = \Omega(B, A) \not\equiv 0$ is holomorphic by the basically weaker assumption that $H'A$ is less than some flat, R_A conformal metric on S. Results similar to Lemma 1 can be obtained, and such generalizations are left to the reader.

Lemma 3. *If* A *and* B *are positive definite,* HA *is complete,* $K(HA) \geqslant 0$ *and* $\Omega = \Omega(B, R_A) \not\equiv 0$ *is holomorphic, then* $K(HA) \equiv K(B) \equiv 0$, $H/H' \equiv c$ *and* $\Omega \neq 0$.

Proof. As in the proof of Lemma 2, R_A is parabolic. Part (vi) of the Main Lemma gives $K(B) \geqslant 0$, while part (iv) provides a superharmonic function $\cosh^{-1}(H/H') > 0$ on R_A except at isolated points where $H' = 0$. But isolated values of $+\infty$ for a superharmonic function are allowed. Thus H/H' is constant, and by the Main Lemma (iv), $K(HA) \equiv K(B) \equiv 0$, while H' and Ω never vanish.

Lemma 4. *If* A *and* B *are positive definite,* B *is complete,* $K(B) \leqslant 0$, H/H' *is bounded and* $\Omega = \Omega(B, R_A) \not\equiv 0$ *is holomorphic, then* $K(A) \equiv K(B) \equiv 0$, $H/H' \equiv c$, *and* $\Omega \neq 0$.

Proof. Since H/H' is bounded and $H > 0$, neither H' nor Ω can vanish. By Lemma 1, $HA = (H/H')H'A$ is complete, and since $H/H' > 1$ is bounded, $H'A$ is complete. But $H'A$ is flat, so that R_A is parabolic, by [23]. Since $K(B) \leqslant 0$, the Main Lemma (iv) provides a subharmonic function H/H' on R_A which is bounded from above. Thus H/H' is constant, and by the Main Lemma (iv), $K(HA) \equiv K(B) \equiv 0$.

Lemma 5. *Suppose that A is positive definite and complete, B is indefinite, and $\Omega = \Omega(B, R_A) \not\equiv 0$ is holomorphic. Then $K(HA) \equiv K(B) \equiv 0$, and H/H' is constant if either $K(B) \geqslant 0$ and $H \leqslant c \leqslant 0$, or $K(B) \leqslant 0$ and $H \geqslant c > 0$.*

Proof. Here, $K < 0$, so that $H' > |H|$. Thus $H' \geqslant c > 0$ if $|H| \geqslant c > 0$, and $H'A$ is complete because A is complete. By the Main Lemma (i), $H'A$ is flat, so that R_A is parabolic, by [23]. If $K(B) \geqslant 0$ and $H < 0$, the Main Lemma (iv) provides a subharmonic function $H/H' \leqslant 0$ on R_A which must be constant. If $K(B) \leqslant 0$ and $H \geqslant c > 0$, apply this reasoning to the pair $A, -B$.

Remark 3. Results of a similar sort can be obtained by reversing the roles of A and B in Lemmas 2 through 5. This task is left to the reader.

Lemmas 2 and 4 can be applied to a harmonic immersion f: $(S, A) \to M^n$ in case $A = II' = X_2'(I, II)$ for the second fundamental form II associated with some choice of a normal vector field. (As shown in [18], if f is harmonic and $A = fI + gII$, then $A\alpha I$ or $A\alpha II'$.) We take for B the metric I induced by the Riemannian metric on M^n, and compute H, K and H' for the fundamental pair I, II. This yields the following. (Theorem 1 is an expanded version of Theorem 6 in [18].)

Theorem 1. *Suppose that f: $(S, II') \to M^n$ is a harmonic imersion and II' is complete. Then $K(II') \equiv 0$ and $H/K \equiv c$ if either $K(II') \geqslant 0$ with $|H/K|$ bounded, or $K(II') \leqslant 0$ with $|K/H|$ bounded.*

Theorem 2. *If f: $(S, II') \to M^n$ is a harmonic immersion and I is complete, with $K(I) \leqslant 0$ and $|H'/H|$ bounded, then $K(I) = K(H'II/K) \equiv 0$, and $H' = cH \neq 0$.*

5. Codazzi pairs and holomorphic quadratic differentials

The results in this section provide an abstract setting for many well known facts. Extensive references are not provided. Some statements (like Theorem 3 and 6) are entirely new. Others (like the lemmas) mix old with new information. Theorem 8 is central to the investigation undertaken in this paper. It provides a coherent context for the examples cited in §3.

The fact in Example 1 was observed by Heinz Hopf, who used it to show that a soap bubble immersed in E^3 which is topologically a sphere must be a sphere. It is widely understood that one can replace the pair I, II in Example

1 by any fundamental Codazzi pair. Thus Hopf's old argument (see [10]) shows that an ovaloid in E^3 with H/K constant must be a sphere, because H/K for the Codazzi pair I, II is H for the Codazzi pair III, II. (See [11].) To describe the exact role of the Codazzi-Mainardi equations in Examples 1, 4 and 5, we have the following.

Lemma 6. *If A, B is a fundamental pair and $C = X_3(A, B)$, then any two of the conditions* (i), (ii) *or* (iii) *imply the third, and* (iv) *as well*:

(i) $\mathrm{Cod}(A, B)$,

(ii) *$H(A, B)$ constant,*

(iii) *$\Omega(B, R_A)$ holomorphic,*

(iv) *$\Omega(C, R_A)$ holomorphic.*

Proof. If $2\partial/\partial z = \partial/\partial x - i\partial/\partial y$ and $2\partial/\partial\bar{z} = \partial/\partial x + i\partial/\partial y$ for any conformal parameter $z = x + iy$ on R_A, the Codazzi-Mainardi equations (12) take on the form

$$2\phi_{\bar{z}} = \lambda H_z, \tag{27}$$

where $\Omega = \Omega(B, R_A) = \phi dz^2$ and $A = \lambda dz d\bar{z}$. The form Ω is holomorphic if and only if $\phi_{\bar{z}} \equiv 0$, while H is constant if and only if $H_z \equiv 0$. Thus any two of the statements (i), (ii) or (iii) implies the third and (iv) as well, since $\Omega(C, R_A) = 2H\Omega(B, R_A)$ by (3).

Remark 4. In Lemma 6, (iv) and (i) give (ii) and (iii) if $K(A, B) \neq 0$, (iv) and (ii) give (i) and (iii) if $H(A, B) \neq 0$, and (iv) and (iii) give (i) and (ii) if $B \not\alpha A$. Analogous comments apply to Lemma 7.

Because $H(HA, B) \equiv 1$ if $H \neq 0$, Lemma 6 has the following consequence.

Theorem 3. *If A, B is a fundamental pair with $H \neq 0$, then* $\mathrm{Cod}(HA, B)$ *is equivalent to $\Omega(B, R_A)$ holomorphic.*

An immersion f: $(S, A) \to M^n$ is harmonic if and only if $\Omega(I, R_A)$ is holomorphic and $\mathcal{H}^* \equiv 0$, where I is the induced metric, and $\mathcal{H}^*$ the A-mean curvature vector field. (See [18].) Theorem 3 gives a more practical characterization of harmonic immersions since both conditions $\mathrm{Cod}(HA, B)$ and $\mathcal{H}^* \equiv 0$ can be checked using arbitrary coordinates on S.

Corollary to Theorem 3. *An immersion f: $(S, A) \to M^n$ is harmonic if and only if* $\mathrm{Cod}(HA, I)$ *and $\mathcal{H}^* \equiv 0$, where $H = H(A, I) \neq 0$.*

By Lemma 6, the condition $\Omega(B, R_A) \not\equiv 0$ holomorphic in Lemmas 2 through 5 can be replaced by the stronger assumption that $\mathrm{Cod}(A, B)$, $H \equiv c$ and $H' \not\equiv 0$. This yields the following, much of which is known. (See [9], [13], [30] or [31].)

Theorem 4. *If A is complete and positive definite with* $\mathrm{Cod}(A, B)$ *and $H \equiv c$, then $K(A) \equiv K(B) \equiv K(C) \equiv 0$, and K is constant in case* (i) *and* (ii) *or* (iii) *holds*:

(i) $K(A) \geqslant 0$, $H' \not\equiv 0$ *and either B positive definite or* H' *bounded*,

(ii) $K(A) \leqslant 0$ *and* $1/H'$ *bounded*,

(iii) *B indefinite and* $cK(B) \leqslant 0$ *with* $c \equiv H \neq 0$.

Proof. Given (i), use Lemmas 2 and 3 together with Lemma 1 and the Main Lemma (vi). That gives H' constant, and since H is constant, so is K. The Main Lemma (ii) gives the rest. Similar arguments check Theorem 4 given (ii) or (iii).

One can apply Theorem 4 to a Codazzi pair C, B. By Fact 6, this yields the following.

Theorem 5. *If A is positive definite,* $C = X_3(A, B)$ *complete,* $\mathrm{Cod}(A, B)$ *and* $H/K \equiv c$, *then* $K(A) \equiv K(B) \equiv K(C) \equiv 0$ *while both H and K are constant in case* (i) *and* (ii) *or* (iii) *holds*:

(i) $H' \not\equiv 0$ *and either* $K(A) \geqslant 0$ *with B positive definite, or* $K(A) \leqslant 0$ *with* $H'/|K|$ *bounded*,

(ii) $|K|/H'$ *bounded and* $K \cdot K(A) \leqslant 0$ *with* $K \neq 0$,

(iii) *B indefinite and* $cK(B) \leqslant 0$ *with* $c \equiv H/K \neq 0$.

Application of Lemma 6 to the pair C, B yields the next result, which shows that the inverse of the Gauss map is harmonic in case H/K is constant on a surface in E^3. Analogous statements based on other lemmas and the observation in Example 4 are left to the reader. Note that $K(C, B) \neq 0$ is automatic when C is nondegenerate for a fundamental pair A, B.

Lemma 7. *If A, B is a fundamental pair with* $C = X_3(A, B)$ *positive definite, then any two of the conditions* (i), (ii) *or* (rii) *imply the third, and* (iv) *as well*:

(i) $\mathrm{Cod}(A, B)$,

(ii) $H(A, B)/K(A, B)$ *constant*,

(iii) $\Omega(B, R_C)$ *holomorphic*,

(iv) $\Omega(A, R_C)$ *holomorphic*.

Lemma 8. *If A, B is a pair with B definite and* $C = X_3(A, B)$, *then any two of the conditions* (i), (ii) *or* (iii) *imply the third, and* (iv) *as well*:

(i) $\mathrm{Cod}(A, B)$,

(ii) $K(A, B)$ *constant*,

(iii) $\Omega(A, R_B)$ *holomorphic*,

(iv) $\Omega(C, R_B)$ *holomorphic*.

Proof. On R_B, the Codazzi-Mainairdi equations (12) take on the form

$$\mu_x = \mu(\Gamma^2_{12} - \Gamma^1_{22}), \quad \mu_y = \mu(\Gamma^1_{12} - \Gamma^2_{11}), \tag{28}$$

where $B = \mu dz d\bar{z}$, and the Christoffel symbols are computed for A. Setting

$D = \det A$, one has

(29) $$\Gamma^1_{11} + \Gamma^2_{12} = D_x/2D, \quad \Gamma^2_{22} + \Gamma^1_{12} = D_y/2D.$$

If $\Omega = \Omega(A, R_B) = \phi dz^2$, the Cauchy-Riemann equations for Re ϕ and Im ϕ are equivalent to

(30) $$\Gamma^1_{11} + \Gamma^1_{22} = \Gamma^2_{11} + \Gamma^2_{22} = 0.$$

Finally, $K = \mu^2/D$ is constant if and only if

(31) $$2\mu_x D - \mu D_x = 2\mu_y D - \mu D_y = 0.$$

Using (29) together with any two of the relations (28), (30) or (31) one gets the third relation. Thus any two of the conditions (i), (ii) or (iii) imply the third, and (iv) as well, because $\Omega(C, R_B) = -K\Omega(A, R_B)$ by (23).

Remark 5. In Lemma 8, $K(A, B) \neq 0$ is automatic. Thus (iv) and (i) give (ii) and (iii), (iv) and (ii) give (i) and (iii), and if $B \not\propto A$, (iv) and (iii) give (i) and (ii). Analogous comments apply to Lemma 9.

When A is definite and $K(A, B) < 0$ is constant, B is indefinite, so that Lemma 8 seems inapplicable. However, $H'(A, B) \neq 0$, and then Lemma 8 applies to the pair A', B'. Since $K(A', B') = K(A, B)$, and Cod(A, B) is equivalent to Cod(A', B'), we have the next result. It gives the new fact that $\Omega(I', R_2')$ is holomorphic on a surface in E^3, S^3 or H^3 with constant negative extrinsic curvature.

Lemma 9. *If A, B is a fundamental pair with $C' = X_3(A', B')$, then any two the conditions* (i), (ii) *or* (iii) *imply the third, and* (iv) *as well*:

(i) Cod(A, B),

(ii) $K(A, B) < 0$ *constant.*

(iii) $\Omega(A', R_{B'})$ *holomorphic.*

(iv) $\Omega(C', R_{B'})$ *holomorphic.*

Remark 6. If $K(A, B) \equiv 0$, Cod(A, B), and A is positive definite, then the form $(1 + H'^2)A + (1 - H'^2)A'$ is flat and positive definite wherever $H' \neq 0$. Further assumptions seem to be required to achieve stronger conclusions.

Because $K(B, K^{\frac{1}{2}}A) \equiv 1$ wherever $K = K(A, B) \neq 0$, Lemma 9 has the following consequence.

Theorem 6. *If A and B are definite,* Cod$(B, K^{\frac{1}{2}}A)$ *if and only if* $\Omega(B, R_A)$ *is holomorphic.*

Putting this together with Lemma 6, one has a slightly different statement.

Corollary to Theorem 6. *Suppose that A and B are definite with* Cod(A, B). *Then H is constant if and only if* Cod$(B, K^{\frac{1}{2}}A)$.

Using Lemma 8 and Remark 3 one proves the following.

Theorem 7. *If A is definite, B positive definite,* Cod(A, B) *and $K \equiv c \neq 0$, then $K(A) \equiv K(B) \equiv K(C) \equiv 0$ and H is constant in case* (i) *or* (ii) *or* (iii) *holds*:

(i) *B complete with either $K(B) \geqslant 0$ and $H' \not\equiv 0$ bounded or $K(B) \leqslant 0$ and $1/H'$ bounded,*

(ii) *$H > 0$, HB complete, $K(HB) \geqslant 0$ and $H' \not\equiv 0$,*

(iii) *$H > 0$, A or C complete, $K(A) \leqslant 0$ and H/H' bounded.*

For those familiar with the classical proof of Hilbert's theorem (see [14]), we note the following relative of Lemma 9.

Lemma 10. *If A, B is a fundamental pair, any two statements below imply the third*:

(i) Cod(A, B),

(ii) *$K(A, B) < 0$ constant,*

(iii) *Asympotitc Tchebychev coordinates are locally available and $B \neq 0$.*

Proof. Classical arguments (see [14]) give (iii) if (i) and (ii) are known. Given the coordinates provided by (iii), the Codazzi-Mainardi equations (12) for $A = \pm(dx^2 + 2\cos\omega\, dxdy + dy^2)$ and $B = 2Mdxdy \neq 0$ read $(\ln|M|)_x = (\ln \sin \omega)_x$ and $(\ln|M|)_y = (\ln \sin \omega)_y$ for $0 < \omega < \pi$. Thus (i) and (iii) together yield $|M| = c \sin \omega$ for some constant $c > 0$, so that (ii) must hold. Similarly, (ii) and (iii) together yield (i).

The next result ties together Lemmas 7 through 10, and gives a common explanation for the facts described in Examples 1, 2 and 3 above.

Theorem 8. *If A is positive definite,* Cod(A, B) *and $\alpha + \beta H + \gamma K \equiv 0$ with $\beta^2 - 4\alpha\gamma \neq 0$, then* (i) *or* (ii) *or* (iii) *must hold*:

(i) *$\gamma = 0$, H is constant and $\Omega(B, R_A)$ is holomorphic.*

(ii) *$\gamma \neq 0$, $\beta^2 - 4\alpha\gamma > 0$ and $\Omega(A, R_X)$ is holomorphic for $X = \beta A + 2\gamma B$.*

(iii) *$\gamma \neq 0$, $\beta^2 - 4\alpha\gamma < 0$ and $\Omega(A', R_{X'})$ is holomorphic for $X' = \beta A' + 2\gamma B'$.*

Proof. If $\gamma = 0$, H is constant and Lemma 6 gives (i). If $\gamma \neq 0$, take $X = \beta A + 2\gamma B$. Since $\alpha + \beta H + \gamma K \equiv 0$,

$$K(A, X) = (2\gamma k_1 + \beta)(2\gamma k_2 + \beta) = \beta^2 - 4\alpha\gamma$$

even at umbilics, where $H^2 = K$. If $\beta^2 - 4\alpha\gamma > 0$, then $K(A, X) > 0$ and X is definite. Thus Lemma 8 gives $\Omega(A, R_X)$ holomorphic. If $\beta^2 - 4\alpha\gamma < 0$, then $K(A, X) < 0$, X is indefinite, and $X' = \beta A' + 2\gamma B'$ is definite, where $A' = X_1'(A, B)$ and $B' = X_2'(A, B)$. Thus Lemma 9 gives $\Omega(A', R_{X'})$ holomorphic.

Hopf's old soap bubble argument used the fact that a holomorphic quadratic differential on a surface homeomorphic to the sphere must vanish identically. Here his argument yields the following statement which provides a unified explanation for classical results which assume either H or K or H/K is constant. (See [11].)

Corollary to Theorem 8. *If A is positive definite,* Cod(A, B) *and* $\alpha + \beta H + \gamma K \equiv 0$ *with* $\beta^2 - 4\alpha\gamma \neq 0$, *and S is topologically a sphere, then A, B is a flat or spherical pair.*

Remark 7. It is well known that one can find a Codazzi pair A, B with $K(A) = K(A, B)$ and $H(A, B) \equiv 1$ which is neither flat or spherical if S is compact with genus $g > 0$. This is why it requires more than compatability conditions to settle the full soap bubble conjecture. One example is formed by identification of corresponding boundary points on a suitable fundamental region for a periodic surface of revolution with $H \equiv 1$ described by Finn in [6]. Choosing $A = I$ and $B = II$, everything fits together smoothly on the surface so formed. Since one can double the size of the initial fundamental region, one can get an example with $H \equiv 1$, and arbitrarily large area with respect to A.

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